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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Thursday 8 November 2012 (morning)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 13]*

A differential equation is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}$, where $x > 0$ and $y > 0$.

(a) Solve this differential equation by separating the variables, giving your answer in the form $y = f(x)$. *[3 marks]*

(b) Solve the same differential equation by using the standard homogeneous substitution $y = vx$. *[4 marks]*

(c) Solve the same differential equation by the use of an integrating factor. *[5 marks]*

(d) If $y = 20$ when $x = 2$, find y when $x = 5$. *[1 mark]*

2. *[Maximum mark: 12]*

Let the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x+y}$, $(x + y \geq 0)$ satisfying the initial conditions $y = 1$ when $x = 1$. Also let $y = c$ when $x = 2$.

(a) Use Euler's method to find an approximation for the value of c, using a step length of $h = 0.1$. Give your answer to four decimal places. *[6 marks]*

You are told that if Euler's method is used with $h = 0.05$ then $c \simeq 2.7921$, if it is used with $h = 0.01$ then $c \simeq 2.8099$ and if it is used with $h = 0.005$ then $c \simeq 2.8121$.

(b) Plot on graph paper, with h on the horizontal axis and the approximation for c on the vertical axis, the four points (one of which you have calculated and three of which have been given). Use a scale of 1 cm $= 0.01$ on both axes. Take the horizontal axis from 0 to 0.12 and the vertical axis from 2.76 to 2.82. *[3 marks]*

(c) Draw, by eye, the straight line that best fits these four points, using a ruler. *[1 mark]*

(d) Use your graph to give the best possible estimate for c, giving your answer to three decimal places. *[2 marks]*

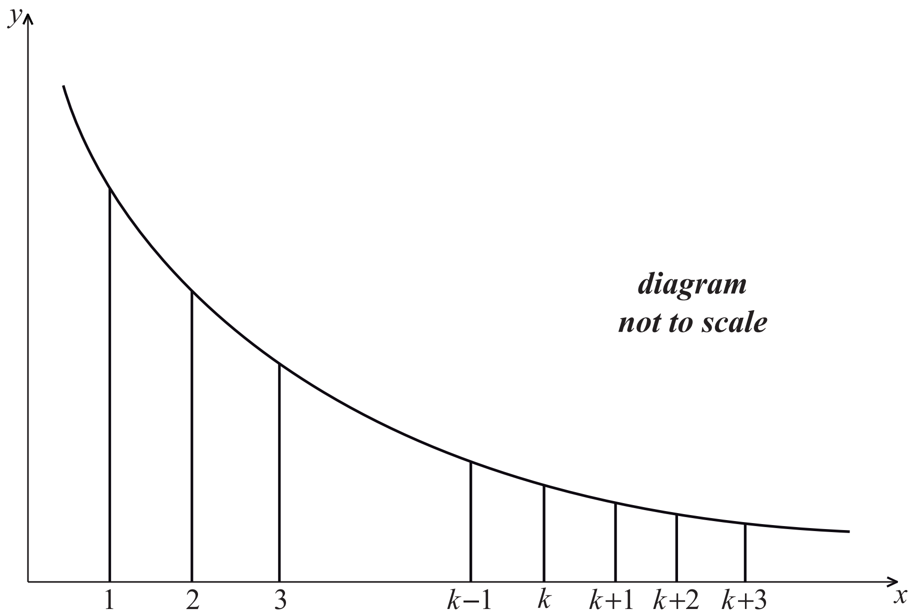
3. *[Maximum mark: 17]*

(a) Prove that $\lim_{H\to\infty}\int_a^H \frac{1}{x^2}\,\mathrm{d}x$ exists and find its value in terms of a (where $a \in \mathbb{R}^+$). *[3 marks]*

(b) Use the integral test to prove that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. *[3 marks]*

Let $\sum_{n=1}^{\infty}\frac{1}{n^2} = L$.

(c) The diagram below shows the graph of $y = \frac{1}{x^2}$.



(i) Shade suitable regions on a copy of the diagram above and show that $\sum_{n=1}^{k}\frac{1}{n^2} + \int_{k+1}^{\infty}\frac{1}{x^2}\,\mathrm{d}x < L$.

(ii) Similarly shade suitable regions on another copy of the diagram above and show that $L < \sum_{n=1}^{k}\frac{1}{n^2} + \int_{k}^{\infty}\frac{1}{x^2}\,\mathrm{d}x$. *[6 marks]*

(d) Hence show that $\sum_{n=1}^{k}\frac{1}{n^2} + \frac{1}{k+1} < L < \sum_{n=1}^{k}\frac{1}{n^2} + \frac{1}{k}$. *[2 marks]*

You are given that $L = \frac{\pi^2}{6}$.

(e) By taking $k = 4$, use the upper bound and lower bound for L to find an upper bound and lower bound for π. Give your bounds to three significant figures. *[3 marks]*

4. *[Maximum mark: 18]*

(a) Use the limit comparison test to prove that $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges. *[5 marks]*

(b) Express $\frac{1}{n(n+1)}$ in partial fractions and hence find the value of $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$. *[4 marks]*

(c) Using the Maclaurin series for $\ln(1+x)$, show that the Maclaurin series for $(1+x)\ln(1+x)$ is $x+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{n(n+1)}$. *[3 marks]*

(d) Hence find $\lim_{x\to -1}(1+x)\ln(1+x)$. *[2 marks]*

(e) Write down $\lim_{x\to 0}x\ln(x)$. *[1 mark]*

(f) Hence find $\lim_{x\to 0}x^x$. *[3 marks]*
